C18. Commission on Mathematical Physics (1981)

Activities

The mandate of C18 covers the mathematical studies of problems originating in or relevant to physics, including mathematical models of physical systems, mathematical aspects of physical theories, and computational techniques.

The current Commission has met twice during IUPAP sponsored conferences, once in London (2000) and once in Paris (2002). Approximately half the membership was present at each meeting. However, most of the discussions among members have been by email and any recommendations or motions have been put to an email vote.

The primary objective of the Commission, during the past three years, has been to find useful ways to serve the community of mathematical physicists and represent its interests to IUPAP. A particular concern has been to promote events in which mathematicians and physicists are able to communicate effectively. With such objectives in mind, we have established strong interactions with the International Association of Mathematical Physicists (IAMP); for example, the President of IAMP was appointed as an Associate Member of C18 and we have frequently consulted with current and past members of the IAMP Executive Committee. As a service to the mathematical physics community, we have developed a C18 web site with links to the various bodies and research institutions in mathematical physics. The site also contains a listing of conferences in mathematical physics. Our hope is that the site will provide a useful resource and help maintain awareness of the many activities in the field.

The C18 Commission has traditionally given its highest priority to supporting and co-sponsoring the triennial International Congress of Mathematical Physics (ICMP) organized by the IAMP. The most recent ICMP was held in London at Imperial College from 17-22 July 2000.

The tradition of co-sponsoring the bi-annual International Colloquium on Group Theoretical Methods in Physics was continued with the 23rd conference in the series held in Dubna, Russia from 31 July to 5 August 2000.

During the current year, C18 joined other IUPAP commissions in co-sponsoring an International Conference on Theoretical Physics held in Paris from 22-27 July 2002. The conference was held in the UNESCO building, which was provided free of charge so that the monies saved could be used to enable many scientists from developing countries to participate.

The next ICMP conference is scheduled to take place at the University of Lisbon from July 28 to August 2, 2003. This conference will be preceded by a two-day symposium for young researchers and several satellite conferences are planned (cf. http://icmp2003.net/). The next International Colloquium on Group Theoretical Methods in Physics will be held in Cocoyoc, Mexico from 2-6 August 2004. A topical conference on the foundations of quantum theory is planned for the centenary of von Neumann's birth in October 2003 in Budapest.

D.J. Rowe, Chair
M.B. Ruskai, Vice-Chair
W.D.Heiss, Secretary

Recent Developments

Unlike other areas of physics, which address particular classes of particles, types of structures or energy domains, mathematical physics does not have clear boundaries. Indeed, the commission has spent some time debating the scope of both mathematical physics and C18. All physics is mathematical, in the sense that physics is a quantitative subject and mathematics is the language of physics. Nevertheless, a distinction can be made between the use of routine mathematical tools and situations in which mathematical analysis or structures provide new physical insights that have an impact on the development of physics. This report gives examples of developments in recent years rather than a comprehensive list.

Theory of Radiation

In the late 1990's, V. Bach, J. Fröhlich and I.M. Sigal considered a non-perturbative mathematical framework for a consistent and rigorous treatment of radiation processes. The goal was to give a mathematical description of the physical phenomena standing at the origin of quantum theory -- emission and absorption of radiation by systems of non-relativistic matter such as atoms and molecules.

The main results of this work are: proofs of the existence of the ground state, of exponential localization of the particle system in this ground state, and of instability of excited states of the particle system and their bifurcation into resonances when the coupling between the particles and radiation is taken into account.

The mathematical theory developed provides an effective method for computation of radiative corrections and, in particular, Lamb shifts, and lifetimes of metastable states. This is done through application of a novel renormalization group technique. The latter uses a renormalization flow that acts directly on a space of quantum equations of motion or quantum Hamiltonians rather than on Green's or correlation functions.

Although the BFS work covers many areas of physical interest, it assumes extremely small values for the fine structure constant and considers only non-relativistic systems. In a related development Lieb, Loss, et al. obtained several results on the self-energy and ground state valid for physically realistic values of the fine structure constant and some models of relativistic mechanics. In the last three years, there has been an explosion of activity, with crucial extensions and improvements of some of the results described above as well as important progress on other aspects of the problem of interaction of quantum radiation with non-relativistic matter.

Field and String theory

Since the 1999 review by Maldacena et al. on the correspondence between String theory and conformally invariant (super) Yang-Mills theory, this so-called AdS/CFT duality, in which quantum and classical properties of for instance quantum anomalies and series expansions in anti-de-Sitter and its conformal boundary are related, has led to considerable results.

In particular departure from conformal invariance is still manageable for integrable perturbations of the quantum theory. Other backgrounds than anti de Sitter are under active study. One of the frontiers is to dispense with supersymmetry by breaking it for instance so as to handle realistic 4 dimensional theories.

Another important physical problem is the regularization of space-time singularities (orbifold singularities) by quantum and string effects.

The picture of a web of dualities some of them exchanging weak and strong couplings, as in the Sine-Gordon Thirring equivalence, is emerging as a key to understanding nonperturbative processes. These dualities are intimately related to properties and the existence of certain arithmetic groups and modular forms that provide exact forms of partition functions and other amplitudes. The key to understanding weak-strong coupling dualities is the presence of certain extended solitonic objects, the so-called D-branes. From a space-time point of view, D-branes arise by imposing Dirichlet-type boundary conditions on the open string fields. Various K-theories have been introduced to handle charges of these D-branes. From a worldsheet point of view, D-branes are described by boundary conformal field theories, and the use of techniques from integrable systems has proven to be quite successful. Dynamical processes involving D-branes, such as creation and annihilation of D-brane anti-D-brane pairs due to tachyon condensation, have led to a renewed interest and remarkable progress in the investigation and reformulation of string field theories.

Mirror symmetry is one other instance of (perturbative) duality. The mathematical study of mirror symmetry, and algebraic geometry in general, Algebraic geometry is expected to be even more useful and to benefit enormously from the supergravity-string theory-M-theory perspective and profusion of concrete and unexpected features.

Finally noncommutative geometry is taking a prominent place in the field. It arises, for instance, in the description of D-branes in the presence of a background two-form gauge field (B-field). In general, noncommutative geometry is expected to be crucial in Quantum Gravity as the notion of a space-time point looses its relevance at short distances. In fact, even the number of dimensions of space-time itself is an artifact of a particular classical limit.

Aside from String Theory, noncommutative geometry and its close cousin, quantum groups, are still very active fields of study. For instance, Hopf algebras have recently been used to handle perturbative quantum field theory expansions and renormalisation (Kreimer and Connes).

Two-dimensional conformal invariance

The challenges of quantum field theory, statistical mechanics, and geometric probability are closely related. A central question is the nature of the phases of a system. As the parameters of a system are varied, the qualitative behavior of the system can change radically as the system makes a transition from one phase to another. At such a critical point there can be special behavior. One insight of recent decades is that this special behavior can display universality: certain numbers (such as critical exponents) have values that are the same in quite different systems. Some explanation of this universality has been provided by the ideas of the renormalization group.

More recently, it has been realized that many two dimensional systems should have a much greater symmetry in the continuum limit. For certain models of geometric probability, such as percolation and self-avoiding walks in two-dimensional lattices, conformal invariance has proved to be a useful tool. Furthermore, new universal geometrical properties emerge. It is possible to predict not only critical exponents, but also various universal probability distributions. Some of these properties are described in elementary terms by the stochastic Löwner evolution with parameter κ, introduced in 2000 by Schramm. The evolution is a random process of growth of a set. It may be constructed from a Bessel process corresponding to dimension parameter d. When d is a natural number, this process is the radial component of a d-dimensional Brownian motion. Moreover, these ideas can be extended to positive real d, and even to complex d. Since $\kappa = 4/(d -1)$, the most regular regime $\kappa \leq 4$ corresponds to $2 \leq d$ when the Bessel process is repelled from the origin.

The picture is that different models correspond to different values of κ. Thus a self-avoiding walk should be described by $\kappa = 8/3$, while percolation should be described by $\kappa = 6$. Recently the conformal invariance of critical site percolation on the triangular lattice in the continuum lattice was proved by Smirnov. His work also provides a rigorous proof that $\kappa = 6$ describes this system, and a verification of Cardy's 1992 formula for crossing probabilities as well as Aizenman's conjecture of their conformal invariance. This is a remarkable vindication of the ideas of conformal invariance, and it applies to probability models that are rigorously defined and intuitively understandable.

Quantum information theory

The current burst of activity in quantum computation and information theory owes much to two mathematical advances of the 1990's as well as experimental achievements. Although much has been made of Shor's factoring algorithm, his subsequent realization that quantum error correction and fault tolerant computation were possible moved the construction of a quantum computer from an unrealistic pipe dream to a formidable, but not unfeasible, challenge for experimentalists. Moreover, the subsequent development of the stabilizer construction of quantum error correcting codes gives yet another example of the way that group theory provides powerful and elegant methods for dealing with mathematical problems that arise in physics.

Historically, proposals for quantum computation grew out of questions about irreversibility and the realization in the 1980's that in quantum mechanics gates would quite naturally be associated with unitary operators that are reversible. The potential power then comes from that fact that the action of a unitary operator on a superposition gives an effective parallelism. It was therefore rather surprising that in 2001 several groups found new methods for universal quantum computation in which the gate action is the result of *irreversible* measurements performed on the computer in a highly entangled state. In addition to being significant in their own right, these developments raise many challenging new mathematical questions in the theory of computation and classification of entanglement.

Quantum information theory also provides new methods for cryptography, communication and such important new applications as digital signatures, which are much more tractable experimentally. The extension of Shannon's classical information theory to this non-commutative arena is non-trivial because of the much richer variety of resources and protocols, leading to new mathematical questions. Many of the necessary tools, such as the strong subadditivity of quantum mechanical entropy and the notion of completely positive maps, were developed by mathematical physicists studying the rigorous foundations of quantum statistical mechanics and the theory of open quantum systems in the 1970's. Among noteworthy recent developments are: the discovery of a simple closed-form expression for the entanglement-assisted classical capacity of a noisy quantum channel, King's use of maximal *Lp* norms to establish additivity of the minimal entropy and Holevo capacity for a large class of channels, and the discovery of a connection between the superadditivity of Holevo capacity and the entanglement of formation.

Group representations, geometric quantization, and coherent states.

The theories of induced representation, geometric quantization, and coherent states are central to the theory of group representations and its applications in quantum physics. Inducing constructions were introduced into physics in a classic paper by Wigner in which he constructed all the irreducible unitary representations of the Poincaré group for positive mass particles. The theory of geometric quantization is the most sophisticated method for quantization of a classical system and solves in an elegant geometrical way the ambiguities of Dirac's so-called canonical quantization. Coherent state theory was originally introduced to give the correspondence between classical and quantum mechanics and continues to be used for this purpose in quantum optics. It was extended to a theory of group representation in the so-called Bargmann-Segal representation of the harmonic oscillator. In recent years, the theory of coherent state representations has been much developed. In its extension to vector-coherent state theory, it is now known to incorporate all the standard inducing constructions, including Mackey's constructions and Harish-Chandra's holomorphic induction. In the last year, it has been shown it to be equivalent to the theory of geometric quantization and to lead to extensions of the latter so that it applies to systems with intrinsic gauge degrees of freedom, such as vorticity.

Thus, the theories of induced representation, geometric quantization, and coherent state representations, are now seen as providing different perspectives on a single inclusive theory. This development is of particular significance for physics, because it brings some relatively sophisticated mathematical methods into practical use for the solution of a wide range of physical problems in the quantization of complex systems with dynamical symmetries. (Reference: S.D. Bartlett et al., J. Phys. A, **35** (2002) 5599-5651.)

Dilute Bose gases

Lieb, Seiringer and Yngvason have obtained a number of results for dilute Bose gases, including a proof of Bose-Einstein condensation in a limit (called the Gross-Pitaevskii limit) appropriate for recent experiments on trapped gases. Furthermore, it has now been shown that a gas with repulsive interactions is 100% superfluid in the dilute limit in which the Gross-Pitaevskii equation is exact. This is the first example of an experimentally realistic continuum model in which superfluidity is rigorously verified.

Chaos in the outer solar system

Classical analytic theories of the solar system indicate that it is stable but numerical integrations suggest that it is chaotic. This disagreement has been resolved in a theory that analyses the effects on the planet Uranus due to the gravitational interaction with Jupiter as it moves in the potential of the sun and Saturn. It shows that the motion of Uranus is chaotic and that at some time in the distant future, due to three-body resonance effects, it could suffer a close encounter with Saturn and be ejected from the solar system. The robustness of the theory has been confirmed by careful numerical integration. It is interesting that Laplace observed such three-body resonances in the solar system many years ago and used them to explain observed non-periodic behavior in the orbits of Jupiter and Saturn. Such resonant behavior has now been shown to produce chaotic motion. It is also noteworthy that a separate source of chaos in the inner solar system, has been revealed by numerical integration. It is not yet understood and so provides fertile ground for future dynamicists. (Reference: N. Murray and M. Holman, Science vol. 283 (1999) 1877-1881.)

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